Test of final state approximations using threshold $pp \to pp\pi^0$

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Abstract

The Watson-Migdal approximation scheme to take into account final state interactions is shown to give the actual threshold momentum dependence of the reaction $pp \to pp\pi^0$. However, by an explicit plane wave replacement of the final state wave function it is stressed that not too much physical significance should be given to the proportionality coefficient extracted using this procedure. The plane wave approximation is not physically reliable even after introducing the Watson-Migdal or a more sophisticated final state interaction factor, since direct production (impulse term) is missed. Also with short range interactions there can be discrepancies of a factor two.

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During the 90's meson production at threshold has raised much experimental activity and interest with cooler facilities producing data with unprecedented accuracy and energy resolution in a region where only a single partial wave amplitude should contribute. One of the exciting results is the threshold cross section of the reaction $pp \to pp\pi^0$ [1,2], others involve e.g. η meson production [3]. This activity has also revived theoretical interest with many mechanisms proposed for each reaction. Discussion of these mechanisms is not the main purpose of this paper.

Most theoretical approaches are based on the established and apparently in this case numerically well justified DWBA including various irreducible production mechanisms between initial and final states calculated for realistic interactions. There is, however, one class of works that use strong assumptions about the possibility to approximate the NN wave functions either by

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using apparently a plane wave Born approximation or including just the low energy final state interaction (FSI) approximately. In the latter it is assumed that the effect of the FSI can be factorized from the transition matrix elements and be included after the matrix elements have been calculated using simple analytic wave functions, notably plane waves. Among such approaches - while having the value of introducing the mechanisms at a fundamentally more basic level as relativistically covariant or by chiral perturbation theory - are Refs. [4,5] on π^0 production and Refs. [6] on η production. The aim of this paper is to test how well such approximations reflect the numerical (and partly physical) reality by an explicit calculation of $pp \to pp\pi^0$.

At low energy or threshold scattering and reactions an often used approximation to take into account the final state interaction is the Watson-Migdal procedure [7]. This consists of taking first the asymptotic scattering (final) state wave function with the relative NN momentum p_f

$$\frac{u_f(r)}{p_f r} = \frac{\sin(p_f r + \delta)\, e^{i\delta}}{p_f r} \tag{1}$$

(only the nucleon S-wave is considered here) and then extending this asymptotic form to the range of nuclear forces (where $p_f r \approx 0$; r cancels against the volume element in integration)

$$\frac{u_f(r)}{p_f} \to \frac{\sin\delta\, e^{i\delta}}{p_f} = \frac{1}{p_f \cot\delta - i p_f}. \tag{2}$$

Here it is usual to make a further approximation in terms of the scattering length by replacing

$$p_f \cot\delta \approx -\frac{1}{a}, \tag{3}$$

where a is the large scattering length. So in S waves one expects the threshold behaviour of the cross section to be

$$\sigma \sim \frac{1}{p_f^2\,(\cot^2\delta + 1)} \times p_f \sim \frac{a^2}{1 + (p_f a)^2} \times p_f, \tag{4}$$

where the p_f in the numerator is the momentum dependence of the phase space. If the infinite ranged Coulomb force is present in the final state the cross section should behave like

$$\sigma = \frac{1}{C_0^2}\frac{1}{p_f^2(\cot^2\delta + 1)} \times \text{phase space} \times \text{const.} \equiv F_{FSI} \times \text{phase space} \times \text{const.}\,, \tag{5}$$

where δ is now the Coulomb-strong phase shift and C_0^2 is the Coulomb penetration factor

$$C_0^2 = \frac{2\pi\eta}{e^{2\pi\eta}-1}, \qquad \eta = \frac{e^2 M_p}{2\hbar^2 p_f}\,. \tag{6}$$

The remaining interaction matrix element is assumed to be relatively constant as compared to the fast varying F_{FSI}. However, it is easy to erroneously imply some direct specific meaning(s) for the constant extracted in this way from experiments. There is nothing in Ref. [7] which would relate the actual reaction matrix element to this constant, since the radial dependence obtained from the asymptotic form is meaningless at short distances. In particular this constant is *not* to be interpreted as the Born approximation to the reaction matrix element and this interpretation is not advocated in the original work. It only reflects the general final momentum or energy dependence embedded in F_{FSI} for small momenta p_f.

To exemplify this in a specific reaction with the strongly attractive nearly bound ${}^1S_0$ two-nucleon final state I discuss very explicitly the reaction $pp \to pp\pi^0$ close to threshold. This has both complications: the strong interaction and the Coulomb interaction in the final state. The energy discussed is 290.7 MeV corresponding to $p_f(\mathrm{max}) = 0.35\,\mathrm{fm}^{-1}$ and is about 5 MeV above threshold in the c.m.s.. The formalism has been given elsewhere [8] and will not be repeated here as not very essential for the present argument. Suffice it to say that the pion is produced primarily by the Galilean invariant πNN vertex for each nucleon obtained from the pseudovector pion coupling

$$H_{\pi NN} = \frac{f}{m_{\pi^+}}\,\vec{\sigma}\cdot\left\{\vec{q}\,\vec{\tau}\cdot\vec{\phi} - \frac{\omega_q}{2M}\left[\vec{p}\,\vec{\tau}\cdot\vec{\phi} + \vec{\tau}\cdot\vec{\phi}\,\vec{p}\right]\right\}. \tag{7}$$

Also it is necessary to include at least pion s-wave rescattering from the second nucleon and some additional mechanism, such as a heavy meson exchange with a nucleon z-diagram in the production vertex to fit the data (without this, direct production is a gross underestimate as compared to data). In reactions where the NN isospin changes this s-wave rescattering even dominates at threshold, but in the present case it is smaller than the direct axial charge term. To avoid unnecessary complications the treatment here considers only nucleons, no Δ components.

The conventional mechanisms lead to the exact transition amplitude at the two-nucleon level

$$<{}^1S_0|H_{\mathrm{prod}}|{}^3P_0> = \frac{-8\pi}{p_f p_i\sqrt{\omega_q}}\frac{f}{m_{\pi^+}}\left\{(1-\frac{\omega_q}{2M})\,q\int u_f^*(r) j_1(qr/2) u_i(r) dr\right.$$

$$+\frac{\omega_q}{M}\int u_f'^*(r) j_0(qr/2) u_i(r) dr$$
$$+\frac{\lambda_1}{m_{\pi^+}}\left[(2-\frac{\omega_q}{2M})\int u_f^*(r) f'(r) j_0(\frac{qr}{2}) u_i(r) dr\right.$$
$$\left.-\frac{\omega_q}{M}\int u_f'^*(r) f(r) j_0(\frac{qr}{2}) u_i(r) dr\right]\Big\}. \quad (8)$$

Here the first term arises from the standard p-wave πN coupling proportional to the pion momentum $\vec{\sigma}\cdot\vec{q}$ and should be small at threshold due to the presence of the pion wave function $j_1(qr/2)$ in the integrand and the factor q. The Galilean invariance term does not have this suppression, just the factor $\omega_q/M < 1$. The last two terms involve s-wave pion rescattering off the second nucleon with the isospin symmetric amplitude λ_1. The Yukawa function $f(r)$ and its derivative $f'(r)$ arise from the propagator of the intermediate pion and have a range rather similar to OPE [9]. The wave function derivative in the present normalization (see Eq. (1)) is $u'(r) = r\,\mathrm{d}(u(r)/r)/\mathrm{d}r$. One may note that the latter terms involve an "interaction" of finite range, while the first two overlaps have an infinite range. It is therefore of particular interest to see how well short range expansion of the final state wave function, the basis of the Watson-Migdal method, fares here as compared with an exact calculation. Of course, correlations are necessary for the reaction to happen at all, since an on-shell nucleon cannot emit a real pion. So, actually the reaction does have a finite range, though the integrals to be evaluated extend to infinity and no convergence factor is present. This expectation will be borne out in numerical results, where little difference is found between the energy dependences of the first and last terms.

The above integrals can be calculated numerically [8]. The dominant terms near threshold are the Galilean invariance term (the second) and the rescattering terms. When the pion has enough energy, the first term becomes comparable (in fact, physically it will need also the inclusion of the Δ). For the present study of the effect of the final state interaction, the dependence on the relative energy (or momentum p_f) of the final state nucleons is particularly relevant. So special care with also the long ranged Coulomb force is important, since it has large influence at low energies. In all cases presented here the initial state is the same distorted $^3P_0$ wave function calculated from the Reid soft core potential [10]. This potential is also used for the final state in the "exact" case.

Fig. 1 shows the ratio of the squared transition matrix element (calculated using the exact wave function) to the final state interaction factor F_{FSI} as a function of the final state relative pp momentum with the inclusion of different mechanisms. The dotted curve shows the small effect of the direct axial current coupling (nongalilean first term, multiplied by 10) $f/m_{\pi^+}\,\vec{\sigma}\cdot\vec{q}$, where $\vec{q}$ is the pion momentum. The dashed curve includes also the much more important axial charge coupling (Galilean part) $-f/m_{\pi^+}\,\omega_q\,\vec{\sigma}\cdot(\vec{p}+\vec{p}')/2M$. In

Fig. 1. The ratio of the squared matrix elements and the final state interaction factor F_{FSI} defined in Eq. (5). Dotted curve: direct axial current contribution (nongalilean) multiplied by 10; Dashed: full direct Galilean production operator $H_{\pi NN}$ used; Solid: full result including also pion s-wave rescattering; Dash-dot: full result with added HME to fit the data at threshold (divided by 10).

the solid curve also pion s-wave rescattering is added to these impulse terms - in several pionic reactions this is sufficient and satisfactory. By far, most of this contributes through the nongalilean part. The minute direct nongalilean result has a qualitatively different energy dependence, since it has to vanish, when $q \to 0$, i.e. when $p_f \to p_{f,max}$.

However, this reaction needs some additional mechanism [1] and in the dash-dot curve a phenomenological heavy meson exchange is added to reproduce the experimental cross section [11]. It can be seen that in this momentum range, apart from the FSI factor, the momentum dependence is moderate for all mechanisms and the validity of the Watson-Migdal procedure is confirmed. It may also be noted that even the "long-ranged" galilean impulse term does have the same momentum dependence as those with an explicit short-range interaction. This is, of course, due to the fact that the correlations necessary for the reaction are generated by meson exchanges. If the Coulomb interaction is removed, the results are nearly the same: the ratios increase by 10-20% and the energy dependence is marginally weaker.

As another step we try to interpret the matrix element as arising simply from a use of a plane wave for the final state, trying to take the FSI into account just by the factor F_{FSI}. Fig. 2a shows the behaviour of the squared transition matrix for the direct (impulse) contribution as a function of the relative final two-nucleon momentum p_f. The solid curve employs the exact final state wave function (likewise in the other figures). To make a "Born approximation" with respect to the final state one replaces $u_f(r)/(p_f r)$ in Eq. (8) by $j_0(p_f r) = \sin(p_f r)/(p_f r)$. The resulting negligibly small approximate results are not shown in the figure.

In fact, both approximate impulse amplitudes are much smaller than the corresponding exact ones, the nongalilean one by two orders of magnitude, the galilean one by three orders. This smallness is apparently due to the overall orthogonality of the three relevant Bessel functions [12] and to the fact that the $^3P_0$ initial wave function does not deviate much from the plane wave (at 290.7 MeV the calculated phase shift is -9.5°). It may be noted that also the

Fig. 2. Momentum dependence of the squared transition matrix. Curves: Solid) exact final state; Dashed) final state approximated by a plane wave and multiplied by F_{FSI} of Eq. (5) (fm^9); Dotted) exact s-wave rescattering without the impulse term; Dash-dot) the FSI factor from Eq. (9) used. Windows: a) direct ("impulse") terms only, b) pion s-wave rescattering included, c) also phenomenological heavy meson exchange added, d) only HME. In a (direct terms) the approximate results are negligible (smaller by 4–6 orders of magnitude).

derivative of the final S-wave as defined above is a Bessel function. Making the NN potential stronger increased these amplitudes; making it zero further reduced the direct amplitudes by another two orders of magnitude acting as a check on numerics of these infinite-ranged oscillatory integrals.

The strong S-wave final state interaction removes this orthogonality making the amplitude sizable. These correlations influence particularly much on the derivative, making the matrix element of the galilean term physically important. Also inclusion of an explicit interaction such as pion rescattering removes the orthogonality in the integrand. Then apparently at threshold the approximate integral is relatively insensitive to p_f, since $j_0(p_f r) \approx 1$ for small r.

With a finite ranged interaction the energy dependences of the approximate and exact results are qualitatively similar (dashed curves vs. solid in Figs. 2b-d), once the FSI factor is applied, as would be obvious from the constancy of the approximate integral. However, physically in the exact result it is necessary to take into account the significant galilean impulse term. In the galilean case (Fig. 2b) the impulse term is constructive with rescattering and the exact cross section is larger than the approximate FSI result, while in the nongalilean result the situation would be reversed. Even the FSI factor cannot reproduce the full height of the exact result, even though the approximate rescattering is more than twice as large as the exact (dotted curve in 2a). This shows clearly the physical importance of the galilean impulse term.

Finally in Fig. 2c the overestimation of short ranged mechanisms by the approximate factor (shown explicitly in 2d) reproduces the exact result (with a slight overestimate). However, this good agreement is achieved by overestimation of the short-ranged interactions and omission of direct production.

Now it is important to note that, since also the approximate final wave function includes the $1/p_f$, it is *not* consistent to apply the FSI factor defined by Eqs. (2–5) where the same $1/p_f$ appears – the result would not be even

dimensionally correct. If this double counting is avoided, the resulting energy dependence clearly would disagree with the exact result - the dashed curves should be multiplied by p_f^2. This exercise is done, because a more sophisticated treatment is not numerically very different (if p_f is given in fm^{-1}) and because it is possible that the simpler method may have been applied some times in the past. Also, it was seen that the energy dependence comes out correctly.

An enhancement factor of the amplitude having a correct high-energy limit is based on properties of the Jost function and presented in Ref. [13] for a short-ranged interaction applying the effective range expansion to order p_f^2

$$\frac{1}{\mathbf{f}(-p_f)} = \frac{(p_f^2+\alpha^2)r_0/2}{1/a + r_0 p_f^2/2 - ip_f} \tag{9}$$

with

$$\alpha = (1+\sqrt{1+2r_0/a}\,)/r_0. \tag{10}$$

Except for the numerator, to this order this is equivalent to the simpler form and is dimensionless. Also, if the unit of length is the femtometer, the numerator is indeed of the order of unity for small p_f. In Ref. [4] this is used with a Coulomb modification on the scattering length $a = 7.8243$ fm and effective range $r_0 = 2.7058$ fm given now as [14]

$$1/a_c = [1/a - 2p_f\eta h(\eta)]/C_0^2 \tag{11}$$

$$r_{0c} = r_0/C_0^2 \tag{12}$$

with the function $h(\eta)$ given on p. 263 of Ref. [13]. However, since a_c and r_{0c} are not constant with p_f, it may not be clear that the assumptions on which Eq. (9) is based are valid in the presence of the long-ranged Coulomb force. The numerical error arising from this complication is likely quite small but not under control.

Now using this improved final state interaction factor the dash-dot curves in Fig. 2 show similar good qualitative energy dependence with the exact result as the dashed ones – only slightly weaker. However, although the normalization is much improved (as can be seen from the dash-dot vs. dotted curve in 2b and dash-dot vs. solid in 2d), still a mere multiplicative factor cannot correct for the missing direct production strength (the dash-dot vs solid curves in 2b and 2c). Consequently, even after the inclusion of the heavy meson exchange the total results remain now as underestimates shown in Fig. 2c, while only this short range contribution is presented in Fig. 2d to see the effect of the range alone. In this case the approximation is a slightly larger overestimate, about a factor of three for the simple factor.

Fig. 3. As Fig. 2 but without the final state Coulomb force

Above, the Coulomb interaction has been included in the final state interaction. It is also interesting to see how the approximation fares without this complication. After all, Eqs. (9-10) are deduced for the uncharged effective range parametrization. One might consider this to describe e.g. $nn \to nn\pi^0$ (hardly physically measurable), $np \to nn\pi^+$ or $np \to np\pi^0$ reactions. However, to avoid superficial differences the kinematics is kept the same relevant to $pp \to pp\pi^0$. Even so, the scattering length and effective range are taken to be the same experimental np singlet parameters $a_{np} = 23.715$ fm and $r_{0np} = 2.73$ fm as in Ref. [4]. Use of the Coulomb modified quantities above relevant to pp scattering would hardly be physically meaningful and would give an incorrect view of the effect.

The results are shown in Fig. 3 in the same way as in Fig. 2. It can be seen that now both approximations are close to each other, since the numerator of Eq. (9) is (incidentally) close to unity, if the momentum is given in fm^{-1}. Now both approximate methods give an overestimate by factor two for matrix elements involving potential ranges. The total result now agrees incidentally with the exact result but due to the overestimate of HME and omission of direct production as earlier for the simpler procedure. Also, it can be seen that for small values of p_f the functional dependence changes qualitatively.

In summary, it was seen that the Watson-Migdal conjecture that at low momenta the reaction matrix momentum dependence can be obtained from the asymptotic scattering wave function (essentially the phase shift) is true in this reaction. However, using this procedure in the other direction by calculating the reaction matrix by plane wave functions and simply applying the final state interaction factor of the Watson-Migdal method or by the method given in Ref. [13] is risky and may lead to physically incorrect conclusions about the reaction mechanisms. At its simplest, the energy dependent enhancement factor F_{FSI} cannot be correctly used with the Born approximation potential matrix elements, whereas the more sophisticated procedure gives them reasonably well. Both approximations overestimate short-ranged mechanisms when the Coulomb force is not present in the final state.

In particular, in $pp \to pp\pi^0$ the multiplicative procedures miss the important direct production mechanism. In principle, it is possible to incorporate most important meson exchanges after the pion production vertex. However, this

may lead to strong violation of unitarity and also obscures the role of the FSI. Inclusion of any FSI factor would then risk doubly counting parts of this interaction.

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$|ME|^2 / F_{FSI}$ (fm^5)
0.
1.
2.
3.
4.
p_f (fm^{-1})
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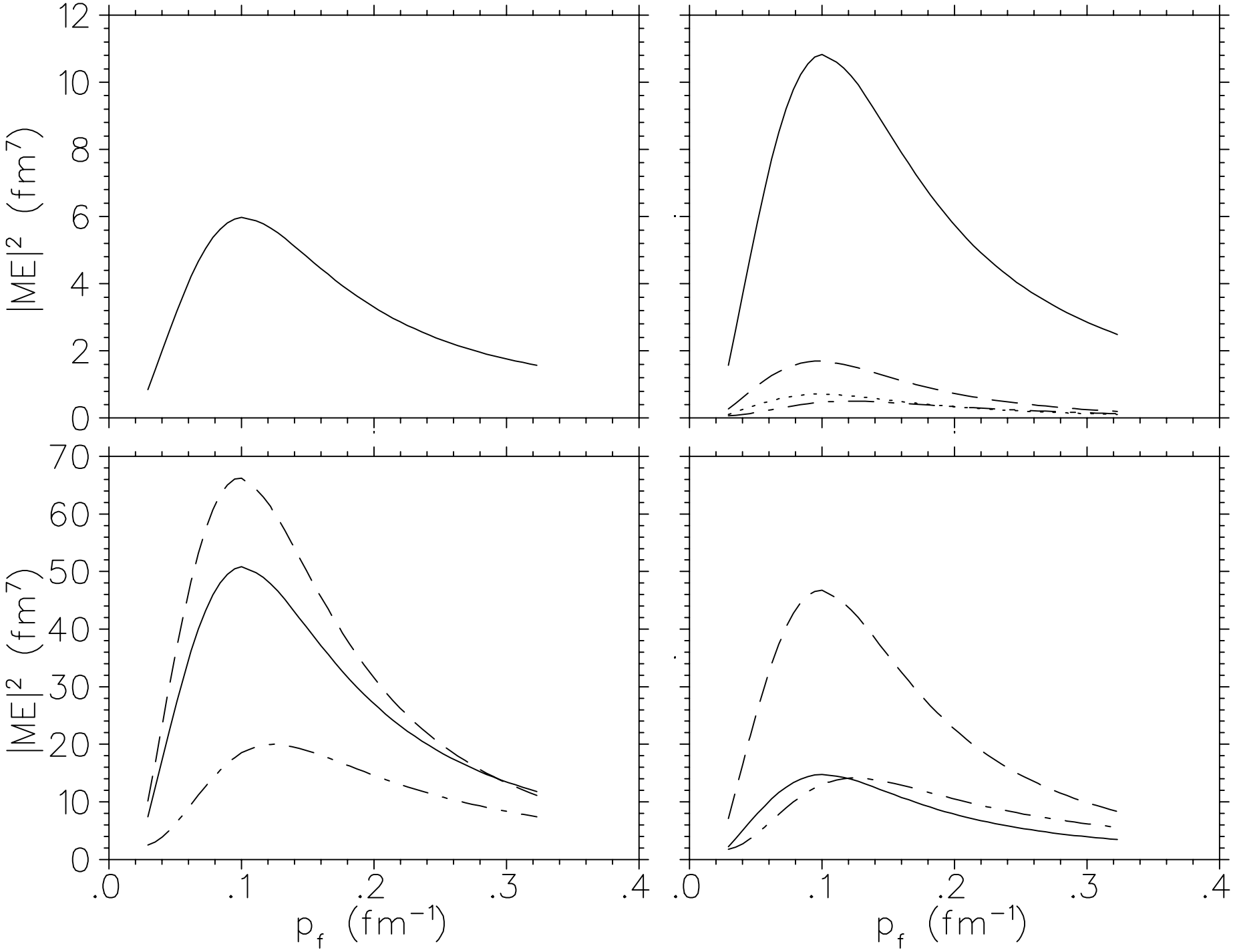
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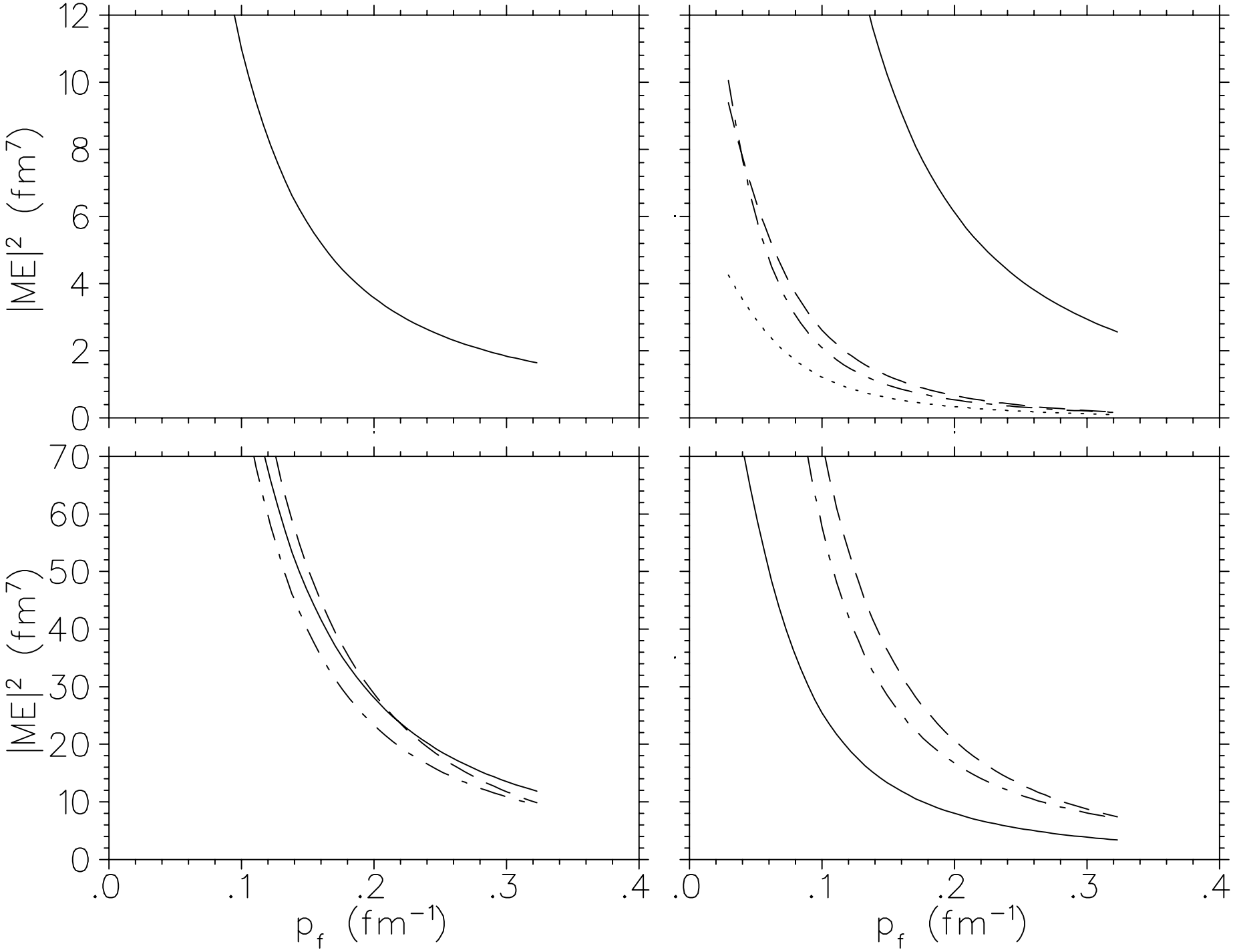
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